Probability density function modeling of dispersed two-phase turbulent flows

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This paper discusses stochastic approaches to dispersed two-phase flow modeling. A general probability density function (PDF) formalism is used since it provides a common and convenient framework to analyze the relations between different formulations. For two-phase flow PDF modeling, a key issue is the choice of the state variables. In a first formulation, they include only the position and velocity of the dispersed particles. The kinetic equation satisfied by the corresponding PDF is derived in a different way using tools from the theory of stochastic differential equations. The final expression is identical to an earlier proposal by Reeks [Phys. Fluids A **4**, 1290 (1992)] obtained with a different method. As the kinetic equation involves the instantaneous fluid velocity sampled along the particle trajectories, it is unclosed. Another, more general, formulation is then presented, where the fluid velocity ''seen'' by the solid particles along their paths is added to the state variables. A diffusion model, where trajectories of the process follow a Langevin type of equation, is proposed for the time evolution equation of the fluid velocity ''seen'' and is discussed. A general PDF formulation that includes both fluid and particle variables, and from which both fluid and particle mean equations can be obtained, is then put forward. [S1063-651X(99)09901-8]

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I. INTRODUCTION

Two-phase flow modeling is a fascinating subject at the crossroads of theoretical considerations and very practical needs. Among the various flow regimes where the geometry of the interface between the two phases differs (for example: annular, slug, or bubble flows), the dispersed flow regime is a particularly important one. Indeed, particulate flows with the dispersed phase present in the form of small spherical solid particles (or liquid droplets) suspended in a gaseous or liquid carrier phase are both of theoretical interest and of considerable practical importance in environmental studies and for numerous industrial processes.

To simulate these flows, the basic field equations must be stated first. A classical hydrodynamical description is assumed for the carrier phase which follows the mass conservation and the Navier-Stokes equations. For the dispersed phase, the starting point is to write the particle equations of motion, but this is less obvious than for the fluid case. Indeed, in spite of numerous studies [1,2], expressing the forces acting on a particle in a general flow is still an open issue. General expressions usually involve the so-called pressure gradient, drag, added mass, and often Basset forces [1,2]. However, for particles heavier than the carrier flow, $\rho_p \gg \rho_f$ with ρ_f and ρ_p the fluid and particle densities, respectively (for example, solid particles or droplets in a gas flow), an acceptable approximation is to retain only the drag force (as well as external forces $\mathbf{F}_e$ such as gravity) in the particle momentum equation which has then the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{V},$$
$$\frac{d\mathbf{V}}{dt} = \frac{\mathbf{U}-\mathbf{V}}{\tau_p} + \mathbf{F}_e, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{V}$ are the particle location and velocity. The drag term is written using the particle aerodynamic relaxation time scale τ_p, and $\mathbf{U}$ stands for the fluid velocity ''seen'' or sampled by the particle as it moves across the flow. In terms of the instantaneous Eulerian velocity field $\mathbf{U}_f$ of the carrier (fluid) phase, this fluid velocity ''seen'' is simply $\mathbf{U}=\mathbf{U}_f(t,\mathbf{x})$. The two phases (the fluid and the particles) exchange momentum and energy. Furthermore, one may have to take particle collisions into account for high-enough concentrations.

One possible way to simulate two-phase flows is therefore to solve the Navier-Stokes equations, to which source terms that represent the exchange of momentum between the particles and the flow may be added. Once the instantaneous velocity field of the carrier flow is known, particles can be advanced since there is no unknown in the particle momentum equation. However, most of the flows encountered in practice are turbulent. They involve a huge number of degrees of freedom, scaling as $\mathrm{Re}^{9/4}$, where the Reynolds number is typically of the order $\mathrm{Re}\sim 10^5-10^7$. Consequently, such a direct approach, in the spirit of DNS (direct numerical simulation), is not feasible in practice and one has to come up with a *reduced or contracted description* that involves far fewer degrees of freedom.

In dispersed turbulent two-phase flow modeling, the first step consists in adopting a statistical point of view, just as in most single-phase turbulence models. This is actually a clas-

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sical step in physics where probabilistic arguments are used in (deterministic) systems involving a very large number of degrees of freedom. This also reflects the fact that we are usually not interested in the full description of the flow and of the particles in space and time but rather in limited information about some statistical characteristics, such as mean velocities, particle mean concentration, and energy, etc. This statistical approach can be performed at different levels of description. The most fundamental level is the direct simulation mentioned above in which all the degrees of freedom (or the exact instantaneous equations) are explicitly solved. This is the equivalent of a microscopic description. At the other extreme of the modeling spectra, one can try to express directly the statistical quantities of interest (mean velocities, mean energies, etc.) as the solutions of partial differential equations. These equations are derived by applying an averaging operator (Reynolds operator or even spatial filtering) to the exact instantaneous equations. This leads to unclosed mean equations; phenomenological assumptions have then to be made (such as the existence of a turbulent viscosity or models for pressure redistribution terms). Such an approach is the equivalent of a macroscopic description. In the present paper, the leading idea is to propose closures at an intermediate level: the exact instantaneous behavior of the system is replaced by a probabilistic model and closures are therefore put forward for the probability density function (PDF) of the state variables which are retained to describe the system. This is called a PDF approach since what is actually modeled is a certain PDF; it can be regarded as a mesoscopic description. This approach is similar to the one already applied to derive the hydrodynamical equations, such as Euler or Navier-Stokes equations. In that context, the hydrodynamical level of description defines the macroscopic level. The equations can either be derived directly by writing balance equations and by assuming Fick and Fourier laws, or by first modeling the PDF equation (typically the Boltzmann equation) at the mesoscopic level from which the hydrodynamical equations are obtained simply as transport equations for different moments of the PDF.

In the following, we limit our attention to dilute two-phase flows where, due to low particle concentration, interparticle collisions and modification of fluid turbulence by the particles can be neglected. However, one central issue, namely particle dispersion, remains to be addressed. This refers to the fact that in the particle equation of motion (1) the driving term, which involves the instantaneous fluid velocity ''seen'' by the particles $\mathbf{U}$, is not known since only limited information (such as the mean fluid velocity $\langle \mathbf{U}_f \rangle$ or its turbulent kinetic energy) is available. This question has already received a great deal of attention in the past years. Most notably, Reeks has proposed an equation, called the kinetic equation, for the PDF of particle location and velocity [3–5]. Apart from this classical PDF approach, a number of stochastic models have been devised to represent the successive fluid velocities encountered by the particles usually under the name of Lagrangian or particle-tracking approaches [6] (cf. [7] for a discussion of popular models).

The present paper has several aims. The first purpose is to propose a general framework, the PDF framework, to analyze and discuss various proposals. This formulation helps to recall and clarify an often-forgotten point: Lagrangian models are also PDF models, however presented directly as a Monte Carlo simulation of an underlying PDF. For that purpose, basic elements of stochastic modeling are briefly presented in Sec. II for the sake of completeness. A second purpose is to put the emphasis on a central issue which is the choice of the state variables retained to describe two-phase flows. This choice is important for many reasons: it defines the amount of information available within one formulation; a different choice makes the modeling step more or less difficult and, above all, more or less justified. This point should not be confused with the precise form of expressions put in the different models. A given model can perform more or less satisfactorily even if the choice of the variables appears ''reasonable.'' Yet, a satisfactory model is probably more difficult to express with an unsuitable choice of state variables. The present work has also some precise aims. In the first part of the paper (Sec. III), a different way to obtain the kinetic equation is presented. The method used here (briefly recalled in the Appendix) is based on the cumulant expansion of the governing stochastic differential equation (SDE) which is the particle equation of motion with a random term. This alternative derivation is believed to be more evident, both mathematically and physically, than previous methods as it avoids the explicit introduction of Kraichnan's LHDIA formalism. In the second part of the present contribution (Sec. IV), a more general formulation is put forward, with the phase space including the fluid velocity along the particle trajectories as an additional variable. A stochastic diffusion model for this variable is written, so that the formulation becomes closed and the governing PDF equation can be derived. Then (Sec. V), a new general one-point PDF is introduced from which both sets of equations (related to both fluid *and* particle mean variables) can be obtained. The discussion is also extended to two-point PDF for the fluid phase since this is the first level of description where the issue of particle dispersion appears in closed form.

II. TRAJECTORY AND PDF APPROACHES TO STOCHASTIC MODELING

Most attempts at turbulence modeling follow a statistical approach, either in terms of transport equations for some mean values (moments) or using the stochastic reasoning. It is the latter that will be of interest for us here (moment equations can always be derived from the closed PDF of a stochastic process). In stochastic modeling, there are two different, yet corresponding, points of view [8]. The first one is the trajectory point of view which consists in writing an evolution equation (generally a stochastic differential equation) for a large number of samples of the vector process $\mathbf{X}$. The second one is the PDF standpoint which deals with the time evolution equation of the PDF P of the process. The equivalence is clearly exemplified for the class of so-called stochastic diffusion processes [8], to which we restrain ourselves in the present paper. The trajectory point of view consists then in writing Langevin types of equations involving the increments of the Wiener processes,

$$dX_i = D_i(\mathbf{X})dt + b_{ij}(\mathbf{X})dW_j, \tag{2}$$

where **D** stands for the drift vector and b_{ij} for the diffusion matrix. In that case, the corresponding PDF equation is the Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i}[D_i(\mathbf{X})P] + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}[B_{ij}(\mathbf{X})P], \qquad (3)$$

where **x** is the phase space variable corresponding to **X**. The Fokker-Planck equation involves the matrix $B_{ij}=b_{ik}b_{jk}$ which is always a positive-definite matrix. Strictly speaking, there is more information contained in the trajectory point of view since different diffusion matrices b_{ij} can give the same matrix B_{ij} in the Fokker-Planck equation. It is only when there is a one-to-one correspondence between the matrices b_{ij} and B_{ij} that the two points of view are strictly equivalent. However, in a loose sense they will be considered below as equivalent.

Models considered in the present paper also fit into this description. The Lagrangian particle models, common in atmospheric dispersion studies, illustrate the trajectory point of view: they consist in a trajectory or Monte-Carlo simulation of the underlying PDF. On the other hand, the model from Sec. III, which we will call (for historical reasons) the kinetic equation model, retains only solid particle locations and velocities and handles a probability density $P(t,\mathbf{x},\mathbf{v})$. The fluid seen here, **U**, is therefore an external force whose statistical effect has to be modeled. On the contrary, in the Lagrangian approach (Sec. IV), this very variable is included in the definition of the system considered and the PDF actually handled is $P(t,\mathbf{x},\mathbf{v},\mathbf{u})$. The point that sometimes gets confused is that the major difference between models lies not in the formulation (either trajectory or PDF), because they are equivalent, but rather in the choice of the vector of state variables.

III. THE KINETIC EQUATION MODEL

In the turbulent two-phase flow domain, pioneering works of Reeks [3,4] established the kinetic equation for dispersed particles and derived the conservation equations next. Analogously to the Boltzmann equation for gas molecules, the kinetic equation for particles is an evolution equation of the probability density $P(\mathbf{x},\mathbf{v},t)$ in the phase space of particle position **x** and velocity **v**. Reeks [3] derived the kinetic equation for particles in homogeneous turbulence, using the random Galilean transformation (RGT). In a subsequent paper [4], the equation has been obtained for the general case of nonhomogeneous turbulent flows. The method used was first developed within the Lagrangian history direct interaction approximation (LHDIA, [9], comprehensively described by McComb [10]). Recently, Hyland [11] presented his alternative derivation of the kinetic equation for dispersed particles, based on results from advanced functional calculus. The resulting equation can be written in the form

$$\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i}\frac{v_i}{\tau_p}\right)P = \frac{\partial}{\partial v_i}\left(\frac{\partial}{\partial v_j}\mu_{ij} + \frac{\partial}{\partial x_j}\lambda_{ij} + \gamma_i\right)P. \qquad (4)$$

Here, $\mu_{ij}(\mathbf{x},\mathbf{v},t)$ and $\lambda_{ij}(\mathbf{x},\mathbf{v},t)$ are diffusion tensors in the phase space and $\gamma_i(\mathbf{x},\mathbf{v},t)$ is a drift vector reflecting the inhomogeneities of the fluid turbulence. They depend on τ_p and are also functionally dependent on the random part of the driving fluid velocity **U** in Eq. (1); the actual form of $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\gamma}$ will be given below.

We will now derive the kinetic equation (4), starting from the system of governing equations (1) for the particle motion in turbulent fluid. To this aim, the method described in the Appendix will be applied directly to Eq. (1), written in shorthand notation as $\dot{\mathbf{Z}}_i = \mathbf{F}_i$. It is equivalent to Eqs. (A3) and (A4) with $\alpha = 1$ and

$$\mathbf{Z} = \begin{bmatrix}\mathbf{x}\\ \mathbf{V}\end{bmatrix},\quad \mathbf{F}^{(0)} = \begin{bmatrix}\mathbf{V}\\ -\dfrac{\mathbf{V}}{\tau_p} + \dfrac{\langle\mathbf{U}\rangle}{\tau_p} + \mathbf{F}_E\end{bmatrix},\quad \mathbf{F}^{(1)} = \begin{bmatrix}\mathbf{0}\\ \mathbf{f}\end{bmatrix}, \qquad (5)$$

where

$$\frac{\langle\mathbf{U}\rangle}{\tau_p} + \mathbf{F}_E = \mathbf{G}.$$

The velocity of the fluid ''seen'' by the particles has been decomposed here into the mean and fluctuation: $\mathbf{U} = \langle\mathbf{U}\rangle + \mathbf{U}'$ and $\mathbf{f} = \mathbf{U}'/\tau_p$.

As shown in the Appendix, from the SDE (A3) for a process **Z**, the corresponding PDF transport equation (A13) can be derived. In order to apply this statement here, a few more expressions present in Eq. (A13) have to be specified and computed first for a particular process **Z**, as defined by Eq. (5).

As already noted in Sec. II, we will adopt throughout the paper the following convention: stochastic processes are denoted by capital letters (**U**,**V**,**Z**, etc.) to distinguish them from the corresponding phase space variables (**u**,**v**,**z**, respectively) or, alternatively, from a particular realization of the process. The unperturbed equation for a deterministic (nonstochastic) system $\mathbf{z}_{(d)}$ writes $\dot{\mathbf{z}}_{(d)} = \mathbf{F}^{(0)}$ or, explicitly,

$$\dot{\mathbf{x}}_{(d)} = \mathbf{v}_{(d)},$$
$$\dot{\mathbf{v}}_{(d)} = -\frac{\mathbf{v}_{(d)}}{\tau_p} + \mathbf{G}. \qquad (6)$$

The subscript ''(d)'' will now be skipped; as explained in the Appendix, we note $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{v}^{-\tau} = \mathbf{v}(t-\tau)$. For $\mathbf{G} = \mathbf{0}$ we obtain

$$\mathbf{v}^{\tau} = \mathbf{v}\,\exp(-\tau/\tau_p),$$
$$\mathbf{x}^{\tau} = \mathbf{x} + \tau_p\mathbf{v}[1-\exp(-\tau/\tau_p)],$$

or (7)

$$\mathbf{v} = \mathbf{v}^{\tau}\,\exp(\tau/\tau_p),$$
$$\mathbf{x} = \mathbf{x}^{\tau} - \tau_p\mathbf{v}^{\tau}[\exp(\tau/\tau_p) - 1].$$

For any function

$$h = h(\mathbf{x}(\mathbf{x}^{-\tau},\mathbf{v}^{-\tau}),\mathbf{v}(\mathbf{x}^{-\tau},\mathbf{v}^{-\tau}))$$

we have

$$\frac{\partial h}{\partial v_i^{-\tau}} = \frac{\partial h}{\partial x_j}\frac{\partial x_j}{\partial v_i^{-\tau}} + \frac{\partial h}{\partial v_j}\frac{\partial v_j}{\partial v_i^{-\tau}}. \qquad (8)$$

Thus, using Eq. (7),

$$\frac{\partial}{\partial v_i^{-\tau}} = \tau_p(1-e^{-\tau/\tau_p})\,\frac{\partial}{\partial x_i} + e^{-\tau/\tau_p}\frac{\partial}{\partial v_i}. \tag{9}$$

Substitution into Eq. (A13), with $s=t-\tau$, results in

$$\begin{aligned}\frac{\partial P}{\partial t} &+ \frac{\partial}{\partial x_i}(v_i P) + \frac{\partial}{\partial v_i}\left(-\frac{v_i P}{\tau_p}\right) \\ &= \frac{\partial}{\partial v_i}\left\{\int_0^t e^{(s-t)/\tau_p}\left\langle f_i(\mathbf{z},t)\frac{\partial}{\partial v_j}f_j(\mathbf{z}^{s-t},s)\right\rangle ds\right. \\ &\quad + \int_0^t \tau_p(1-e^{(s-t)/\tau_p}) \\ &\quad \left.\times\left\langle f_i(\mathbf{z},t)\frac{\partial}{\partial x_j}f_j(\mathbf{z}^{s-t},s)\right\rangle ds\right\} P. \end{aligned} \tag{10}$$

We recall that in the notation of LHDIA [cf. Eq. (A8)]

$$\mathbf{f}(\mathbf{z}^{s-t},s)\equiv\mathbf{f}(\mathbf{x},\mathbf{v},t|s)$$

and **f** should mean the generalized Lagrangian force acting at time s on the particle that passes through **x** with a velocity **v** at some labeling time t. The force depends on the particle trajectory (and this somehow explains the words ''Lagrangian history''); it is, obviously, proportional to the velocity of the fluid ''seen'' by the particle.

If t is considered as the initial time, **f** is the ''classical'' Lagrangian force. On the other hand, for $s=t$ we identify **f** as the Eulerian force and note

$$\mathbf{f}(\mathbf{z}^0,t)\equiv\mathbf{f}(\mathbf{x},t).$$

Thus the final form of the kinetic equation for homogeneous turbulence is obtained and Eq. (10) becomes identical with formula (41) of Reeks [4]:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i}\frac{v_i}{\tau_p}\right)P(\mathbf{x},\mathbf{v},t) \\ = \frac{\partial}{\partial v_i}\left\{\int_0^t e^{(s-t)/\tau_p}\left\langle f_i(\mathbf{x},t)\frac{\partial}{\partial v_j}f_j(\mathbf{x},\mathbf{v},t|s)\right\rangle ds\right. \\ \left. + \int_0^t \tau_p(1-e^{(s-t)/\tau_p})\left\langle f_i(\mathbf{x},t)\frac{\partial}{\partial x_j}f_j(\mathbf{x},\mathbf{v},t|s)\right\rangle ds\right\} \\ \times P(\mathbf{x},\mathbf{v},t). \end{aligned} \tag{11}$$

In the general case of nonuniform flows, the unperturbed equation takes the form

$$\begin{aligned}\dot{\mathbf{x}}_{(d)} &= \mathbf{v}_{(d)}, \\ \dot{\mathbf{v}}_{(d)} &= -\frac{1}{\tau_p(\mathbf{x}_{(d)},t)}\mathbf{v}_{(d)} + \mathbf{G}(\mathbf{x}_{(d)},t). \end{aligned} \tag{12}$$

So, unperturbed particle trajectories are more complicated than those given explicitly in Eq. (7). Following our standard notation, for $\tau=t-s$ they are symbolically written as

$$\begin{aligned}\mathbf{x}^\tau &= \mathbf{x}_{(d)}(\mathbf{x},\mathbf{v},t|s), \\ \mathbf{v}^\tau &= \mathbf{v}_{(d)}(\mathbf{x},\mathbf{v},t|s), \end{aligned} \tag{13}$$

and should be read as the position (velocity, respectively) at time s of a particle that passes through $(\mathbf{x},\mathbf{v})$ at time t. We define

$$g_{ij}(s|t) = \frac{\partial x_{(d)j}}{\partial v_i^{-\tau}}, \quad \dot{g}_{ij}(s|t) = \frac{d}{dt}g_{ij}(s|t). \tag{14}$$

So, instead of Eq. (9) we have

$$\frac{\partial}{\partial v_i^{s-t}} = g_{ij}(s|t)\frac{\partial}{\partial x_j} + \dot{g}_{ij}(s|t)\frac{\partial}{\partial v_j}. \tag{15}$$

Then, following the same steps as in the derivation of the PDF transport equation in the previous case [Eq. (11)], the corresponding PDF equation for dispersed particles in general nonuniform flows, identical to Eq. (88) in the Reeks paper [4], is obtained:

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} - \frac{\partial}{\partial v_i}\frac{v_i}{\tau_p}\right)P(\mathbf{x},\mathbf{v},t) \\ = \frac{\partial}{\partial v_i}\left\{\int_0^t \dot{g}_{ik}(s|t)\left\langle f_k(\mathbf{x},t)\frac{\partial}{\partial v_j}f_j(\mathbf{x},\mathbf{v},t|s)\right\rangle ds\right. \\ \left. + \int_0^t g_{ik}(s|t)\left\langle f_k(\mathbf{x},t)\frac{\partial}{\partial x_j}f_j(\mathbf{x},\mathbf{v},t|s)\right\rangle ds\right\} P(\mathbf{x},\mathbf{v},t). \end{aligned} \tag{16}$$

Comparing the above equation with the shorthand form (4), explicit expressions for diffusion tensors $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$, and for $\boldsymbol{\gamma}$ can be easily found.

Contrary to the Boltzmann equation for gas, where the only mechanism to change the velocity of molecules is via the collision term and the interactions between molecules are supposed to be instantaneous (with no history), in the case of the kinetic equation for particles in turbulent flow, history terms are present. Roughly speaking, these terms are the time integrals over the correlations of fluid velocity along the particle trajectory (fluid ''seen'' by particles).

Although formally the above equation is akin to the Fokker-Planck equation, in fact it is not. It has been put forward [3] that the kinetic equation differs from a classical Fokker-Planck equation since in the particle equation of motion the fluid velocity ''seen'' could not be regarded, in general, as a white-noise term. However, this single argument is not sufficient to rule out similarity with Fokker-Planck equations. If we skip the issue of the modeling of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ and only regard them as given functions, it appears that with the proposed closure expression the resulting kinetic equation does have the form of a general Fokker-Planck equation. Indeed, using vector notation for the state vector, here $\mathbf{X}=(\mathbf{x},\mathbf{V})$, and considering constant λ_{ij} and μ_{ij} (for the sake of simplicity), the kinetic equation is easily rewritten in the form (3) with **B** expressed as a block matrix:

$$\mathbf{B}=\begin{pmatrix} 0 & | & \lambda_{ij} \\ -- & & -- \\ \lambda_{ji} & | & \mu_{ij}+\mu_{ji} \end{pmatrix}. \tag{17}$$

The diffusion matrix which enters the second-order derivative in the Fokker-Planck equation has to be positive definite. However, in the present case, it is straightforward to show that its determinant is $\det \mathbf{B}=-(\det \boldsymbol{\lambda})^2$. This implies that the product of the eigenvalues of $\mathbf{B}$ is negative and that $\mathbf{B}$ has always at least one negative eigenvalue. Therefore, it is not simply the very existence of the tensor $\boldsymbol{\lambda}$ but rather the fact that it causes the $\mathbf{B}$ matrix not to be positive-definite that makes the kinetic equation different from classical Fokker-Planck equations. This also means that the vector stochastic process $\mathbf{x}(t)$, $\mathbf{V}(t)$ is not Markovian. Moreover, since the eigenvalues of $\mathbf{B}$ correspond to diffusion coefficients, the existence of a negative eigenvalue means that, in the phase space of $(\mathbf{x},\mathbf{V})$, the effect of the fluid velocity seen by particles is to induce an ''antidiffusion'' behavior.

IV. THE LANGEVIN EQUATION MODEL

The kinetic equation derived in the preceding section can in principle be used to obtain transport equations for the moments of the PDF, such as the mean particle velocity field, the particle turbulent kinetic energy, etc. It is developed following the PDF point of view and retains only particle variables $(\mathbf{x},\mathbf{V})$ in the state vector. Consequently, information on the statistics of the fluid velocity seen by particles, $\mathbf{U}$, has to be input since this variable is external to the system considered. In the simplest case of homogeneous turbulence, the statistics of the random force can tentatively be assumed known. In particular, the Lagrangian autocorrelation of the fluid velocity along the particle trajectories can be taken as the decaying exponential with the modified integral correlation time (cf. below). However, in the general case, coming up with a satisfactory closure for the flux induced by the fluid seen appears as a difficult task.

An alternative approach to the two-phase system described is to include the fluid velocity along the particle trajectories as a new independent variable. The state vector is then extended to $\mathbf{X}=(\mathbf{x},\mathbf{V},\mathbf{U})$. The new variable is governed by its own evolution equation, and the complete system of time evolution equations becomes

$$\frac{d\mathbf{x}}{dt}=\mathbf{V}, \tag{18}$$

$$\frac{d\mathbf{V}}{dt}=\frac{\mathbf{U}-\mathbf{V}}{\tau_p}, \tag{19}$$

$$\frac{d\mathbf{U}}{dt}=\mathbf{A}. \tag{20}$$

In the last equation, the time rate of change of the fluid velocity seen, say $\mathbf{A}$, is an external term which has to be modeled. The corresponding (unclosed) equation for the PDF $P(t,\mathbf{x},\mathbf{v},\mathbf{u})$ is

$$\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}=-\frac{\partial}{\partial v_i}\left[\left(\frac{u_i-v_i}{\tau_p}\right)P\right]-\frac{\partial}{\partial u_i}[\langle A_i|(\mathbf{x},\mathbf{v},\mathbf{u})\rangle P]. \tag{21}$$

Since the fluid velocity seen is now included in the state vector, the particle momentum equation is closed. However, the closure issue has been shifted to the change rate of the variable $\mathbf{U}$, as manifested by the unknown term $\langle A_i|(\mathbf{x},\mathbf{v},\mathbf{u})\rangle$ in the PDF equation; it denotes the mean value of A_i at $\mathbf{x}$, conditioned on $\mathbf{V}=\mathbf{v}$ and $\mathbf{U}=\mathbf{u}$. This procedure could be repeated any number of times and is typical of a hierarchy of (unclosed) PDF equations such as the BBGKY equations in statistical mechanics. There is, however, a precise physical reasoning behind such a method that can justify such a move [12]. In a given situation, one introduces typical (or reference) length and time scales which, roughly speaking, define the scales or levels at which a system is studied. Various degrees of freedom of the system (possibly an infinite number of them) are then divided into slow and fast variables with respect to these scales (a fast variable is a random variable whose characteristic time scale is much smaller than the reference one). The main idea is to retain in the state vector the slow modes while removing the fast ones (using, for example, fast-variable elimination techniques [13]) and replace them by models which represent their equilibrium values and usually involve white-noise terms. This procedure is successful when the so-called ''fast modes'' have characteristic time scales which are negligible with respect to the reference time scale, thereby justifying replacing their effects by Wiener processes. As an illustration of this procedure, application of Kolmogorov's hypotheses suggests that, in high-Reynolds turbulent flows and for a reference time scale which belongs to the inertia range, fluid particle accelerations are nearly uncorrelated while fluid velocities are still well correlated [14]. Therefore, using the present terminology, Kolmogorov theory indicates that, for a time interval in the inertia range, fluid particle accelerations are fast variables that can be eliminated. In other words, Kolmogorov theory supports the idea of keeping fluid velocity in the vector of state variables and of modeling fluid particle acceleration. This approximation is the starting point behind Langevin equations [12,15] proposed for fluid particle velocities $\mathbf{U}_f$ (denoted by the subscript f to distinguish them from the fluid velocities seen by solid particles). They are developed along the trajectory point of view and the time evolution equations are SDEs. The model takes the form of a diffusion process with a linear drift term [15]

$$dU_{f,i}=-\frac{1}{\rho_f}\frac{\partial\langle p\rangle}{\partial x_i}dt+G_{ij}(U_{f,j}-\langle U_{f,j}\rangle)dt+\sqrt{C_0\langle\epsilon\rangle}dW_i. \tag{22}$$

Here, $\langle p\rangle$ is the mean pressure field, $\langle\epsilon\rangle$ is the mean turbulent energy dissipation rate, and $d\mathbf{W}$ stands for a vector of independent Gaussian white noise. This form of the model can be directly assumed or can be derived from underlying (''microscopic'') modeling steps which make use of Onsager's hypotheses [16]. The particular model we will consider, mainly for simplicity reasons, uses an isotropic form with a return-to-equilibrium term for the matrix $\mathbf{G}$ (Ref. [17]):

$$G_{ij}=-\frac{1}{T_L}\delta_{ij}=-\left(\frac{1}{2}+\frac{3}{4}C_0\right)\frac{\langle\epsilon\rangle}{k}\delta_{ij}, \tag{23}$$

where k is the turbulent kinetic energy.

For two-phase flow modeling purposes, the problem is more complicated since one has to write a model for the velocity of the fluid seen, $\mathbf{U}$, whose statistics differ from fluid particle velocities $\mathbf{U}_f$. The issue is further compounded by particle inertia and by crossing-trajectory effects. Nevertheless, a current and simple way is to use similar ideas and to propose a Langevin model for $\mathbf{U}$ [7,18,19]. Still relying on Kolmogorov hypotheses, a simple model consists in assuming a form similar to the fluid particle case and to write (no sum over i)

$$dU_i=-\frac{1}{\rho_f}\frac{\partial\langle p\rangle}{\partial x_i}dt-\frac{U_i-\langle U_i\rangle}{T^*_{L,i}}dt+\sqrt{B_i}dW_i, \tag{24}$$

where B_i is the diffusion term and T^*_L is the integral time scale of the fluid seen. Csanady's expressions [7] can be used for the time scales to account for the crossing-trajectory effect when a mean drift of particles (due, for example, to an external field $\mathbf{F}_e$ such as gravity) is present

$$T^*_{L,\parallel}=\frac{T_L}{\sqrt{1+\beta^2\xi^2}},\quad T^*_{L,\perp}=\frac{T_L}{\sqrt{1+4\beta^2\xi^2}}. \tag{25}$$

ξ is the normalized drift velocity and β is the ratio of Lagrangian to Eulerian time scales,

$$\xi^2=\frac{|\langle\mathbf{U}\rangle-\langle\mathbf{V}\rangle|^2}{2k/3},\quad \beta=\frac{T_L}{T_E}. \tag{26}$$

The corrections to the (isotropic) fluid time scale, T_L, are different in the direction parallel to the mean drift between the particles and the fluid (noted with the index $\parallel$) and in the direction perpendicular to it (noted with the index $\perp$). When the reference system is chosen with one direction aligned to the mean drift, $i=1$ say, then $T^*_{L,i}=T^*_{L,\parallel}$ for $i=1$ and $T^*_{L,i}=T^*_{L,\perp}$ for $i=2,3$. Therefore, even with the simplest fluid particle Langevin model which involves only one time scale T_L, the extension to two-phase flows requires already a nonisotropic form of the return-to-equilibrium term $G^*_{ij}=-1/T^*_{L,i}\delta_{ij}$.

The value of the diffusion coefficients B_i can be obtained as follows. Let b_i denote the denominator in Eq. (25), i.e., $b_i=T_L/T^*_{L,i}$. The turbulent kinetic energy of the fluid seen is defined as

$$k=\frac{1}{2}\langle(\mathbf{U}-\langle\mathbf{U}\rangle)^2\rangle.$$

In the case of isotropic turbulence, using the Ito formula, from Eq. (24) we can establish the corresponding evolution equation for k and use the identity $dk/dt=-\langle\epsilon\rangle$ which assumes that the time evolution of the kinetic energy of the fluid seen is the same as that of the fluid kinetic energy (we assume here no bias). Finally, this results in the expression for B_i:

$$B_i=\frac{2}{3}\langle\epsilon\rangle\left[\left(1+\frac{3}{2}C_0\right)b_i-1\right].$$

In the case of small-inertia particles ($b_i=1$), B_i becomes equal to $\sqrt{C_0\langle\epsilon\rangle}$ and the evolution of the fluid seen, Eq. (24), becomes identical with that of the fluid, Eq. (22), as expected. Yet, in the presence of a mean drift between the fluid and the particles ($b_i\neq1$), it is worth noting that the diffusion term in the equation for the fluid seen has now an anisotropic expression contrary to the fluid particle case.

The corresponding PDF equation for $P(t,\mathbf{x},\mathbf{v},\mathbf{u})$ is closed and has the form

$$\frac{\partial P}{\partial t}+v_i\frac{\partial P}{\partial x_i}=-\frac{\partial}{\partial v_i}\left[\left(\frac{u_i-v_i}{\tau_p}\right)P\right]+\frac{1}{\rho_f}\frac{\partial\langle p\rangle}{\partial x_i}\frac{\partial P}{\partial u_i}+\frac{\partial}{\partial u_i}\left[\left(\frac{u_i-\langle U_i\rangle}{T^*_L}\right)P\right]+\frac{1}{2}\frac{\partial^2}{\partial u_i^2}[B_iP]. \tag{27}$$

The Langevin equation (24) is certainly not definitive for the general nonhomogeneous case where one expects spatial gradients of turbulence statistics to enter into the picture. The above derivation of the diffusion coefficients can still be carried out but becomes more involved and this extension is not included here. Indeed, the point of the present discussion is not to go into details of various proposals which belong to the same type but rather to compare characteristics of different types of models. Therefore, the simple case of isotropic turbulence is only considered to avoid more complicated forms. Nevertheless, it should be emphasized that, although some proposals have been discussed and used at length [19], they still lack rigorous theoretical justification. They are simply direct extensions of models for real fluid particles. Even if one accepts a Langevin equation for $\mathbf{U}_f$, there is at present no theoretical derivation of a similar Langevin equation for $\mathbf{U}$. In other words, improvement of current models is still very much an important issue and accurate modeling of the fluid velocity seen remains an open question.

V. FLUID AND SOLID PARTICLE PDF PICTURE

The PDF transport equations for the Langevin equation model (27) and the kinetic equation (16) differ by the use of an extra variable. However, both PDF concern only variables attached to solid particles and consequently only statistical properties of the solid phase can be extracted from these PDF. Characteristics of the fluid phase remain external and have to be developed by another route (usually classical Reynolds-stress modeling). The discussion about the choice of the variables suggests extending the PDF framework to include the two phases. Indeed, we are not dealing only with solid particles being randomly carried about by fluid turbulence but with a two-component system. Therefore, it seems a logical step to introduce a fluid-solid particle PDF picture and to discuss fluid and solid properties from the same point of view. The corresponding PDF that is needed for the purpose is written as $P_{fps}(t,\mathbf{x}_f,\mathbf{u}_f;\mathbf{x},\mathbf{v},\mathbf{u})$. It represents the probability that, at time t, a fluid particle takes at location $\mathbf{x}_f$ a velocity $\mathbf{u}_f$ while a solid particle at location $\mathbf{x}$ has a velocity $\mathbf{v}$ and samples a fluid velocity equal to $\mathbf{u}$. It is necessary

to introduce two different independent location variables, namely $\mathbf{x}_f$ and $\mathbf{x}$, since fluid and solid particles are not convected with the same velocities. The indexes of P_{fps} stand for fluid, particle, and sampled (or ''seen'') fluid, respectively.

Two marginal PDF have then a clear meaning and correspond to known proposals. The first one is obtained by integrating over all solid-phase characteristics and is the PDF related to fluid-phase characteristics, noted P_f:

$$P_f(t,\mathbf{x}_f,\mathbf{u}_f)=\int P_{fps}(t,\mathbf{x}_f,\mathbf{u}_f;\mathbf{x},\mathbf{v},\mathbf{u})d\mathbf{x}d\mathbf{v}d\mathbf{u}. \quad (28)$$

The second marginal PDF is obtained by integrating over all fluid-phase characteristics and is the PDF related to solid-phase characteristics (the same as in Sec. IV), noted P_p here:

$$P_p(t,\mathbf{x},\mathbf{v},\mathbf{u})=\int P_{fps}(t,\mathbf{x}_f,\mathbf{u}_f;\mathbf{x},\mathbf{v},\mathbf{u})d\mathbf{x}_f d\mathbf{u}_f. \quad (29)$$

Computations of the complete fluid-particle PDF P_{fps} can be performed using the trajectory point of view or, in other words, by Lagrangian-Lagrangian simulations. Time evolution equations are then written for an ensemble made up of fluid and solid particles which are tracked together. Both have specified variables attached to them which appear as independent variables in the PDF P_{fps}. Fluid particle time evolution equations can be modeled as

$$d\mathbf{x}_f=\mathbf{U}_f dt, \quad (30)$$

$$d\mathbf{U}_f=-\frac{1}{\rho}\nabla\langle p\rangle dt-\mathbf{G}(\mathbf{U}_f-\langle\mathbf{U}_f\rangle)dt+\sqrt{C_0\langle\epsilon\rangle}d\mathbf{W}. \quad (31)$$

This indicates that for fluid-phase characteristics the model is identical to Pope's model [15]. Relations with classical moment equations and interest in this model have already been discussed [17]. The time evolution equations for the variables attached to solid particles are Eqs. (18), (19), and (24). This means that for solid-phase characteristics the model is identical to the one discussed just before. Use of this fluid-particle PDF allows an equal treatment of both phases and is a compact way to present a complete two-phase model. Yet, it neither solves nor simplifies the difficulties related to the modeling of the fluid velocity seen. The necessity of such a model, even for the fluid-solid particle PDF P_{fps}, is not an inherent element of the PDF framework but stems from a limitation to one-point PDF. For two-phase flow problems, the choice of only one-point PDF implies insufficient available information at the particle level. It is worth emphasizing that, if a general *two-point* PDF model were available, the problem of determining solid particle statistical properties would be closed. Indeed, in this case the model for the fluid velocity ''seen'' $\mathbf{U}$ would no longer be needed, since the fluid velocity at the particle location $\mathbf{x}_2$ at t_2, given the particle location $\mathbf{x}_1$ at t_1, could be determined directly from the conditional probability $P_f(t_2,\mathbf{x}_2,\mathbf{u}_2|t_1,\mathbf{x}_1,\mathbf{u}_1)$. This is clearly an indication that, more than improved kinetic closures, the real issue is a multipoint PDF or statistical treatment of the fluid phase.

VI. CONCLUSION

The purpose of the present paper is to discuss the Lagrangian modeling of two-phase dispersed turbulent flows. The important issue for modeling is the choice of the variables that enter the state vector to represent a particular physical system. In the first formulation presented above (the kinetic equation) only solid particle characteristics were considered while the Langevin model includes the fluid velocity ''seen'' as a further and independent variable. It has been recalled that accurate modeling of the fluid seen remains an open issue.

Concerning the first formulation in terms of particle location and velocity, we have proposed a new, concise, and hopefully elegant way to derive formally the kinetic equation for particles in a turbulent flow. The equation is not closed, because of the unknown form of the correlation of fluid velocity along particle trajectories.

In the second formulation, velocity of the fluid seen by particles has been added to the system. This velocity is proposed to be governed by the Langevin equation model; this represents an extension of ideas already used for turbulence modeling. Alternatively, the Langevin model can be looked at as a PDF kind of closure. Therefore, the discussion was carried out using the PDF formalism. Reformulation in terms of PDF leads quite naturally to the theoretical considerations of consistency relations between different closure proposals. Moreover, once a general model for the correlations of fluid velocity ''seen'' by the particles is proposed and validated, the governing Eulerian (i.e., two-fluid) equations for the two-phase flow can be derived from the PDF equation.

Finally, it is suggested not to limit oneself to variables related to solid particles only and to extend the PDF formalism to include quantities of both phases. A first proposal for a fluid-solid particle PDF has been presented.

APPENDIX: FROM SDE TO PDF TRANSPORT EQUATION

This appendix describes a general method to obtain a governing transport equation for the PDF of a stochastic process, given a stochastic differential equation for its trajectories. The mathematical formalism presented below is taken basically from Van Kampen [20]; technical details are given in [21].

Consider first a linear stochastic differential equation (SDE) for a vector process $\mathbf{Z}$:

$$\frac{d\mathbf{Z}}{dt}=[A_0+\alpha A_1(t)]\mathbf{Z}, \quad (A1)$$

where A_0, A_1 are linear operators (they can be thought of as matrices or differential operators); A_0 is deterministic while A_1 is random with a finite autocorrelation time τ_c; α represents the level of fluctuations; it is supposed that $\alpha\tau_c \ll 1$. Using the substitution $\mathbf{Z}(t)=\exp(tA_0)\mathbf{V}(t)$, applying the cumulant expansion [20], limited to the second order in α, taking the ensemble average of the solution of the corresponding SDE for $\mathbf{V}$, and substituting back for $\mathbf{Z}$, results in

$$\frac{d\langle \mathbf{Z}(t)\rangle}{dt}=\Big[A_0+\alpha\langle A_1(t)\rangle + \alpha^2\int_0^t \langle\langle A_1(t)e^{\tau A_0}A_1(t-\tau)\rangle\rangle e^{-\tau A_0}d\tau\Big]\langle \mathbf{Z}(t)\rangle. \tag{A2}$$

The symbol e^B, where B is an operator, is formally defined as the sum of the Taylor series with powers of B and double angular brackets stand for the central moments; for example, a symbol

$$\langle\langle ab\rangle\rangle=\langle (a-\langle a\rangle)(b-\langle b\rangle)\rangle$$

denotes the covariance of two random variables a, b. Equation (A2) can be thought of as a ''renormalized'' form of the initial equation, Eq. (A1), where the effect of fluctuations appears in the form of an additional deterministic operator. The expansion limited to first order in α represents the deterministic ordinary differential equation (ODE) with no effect of random fluctuations.

For a nonlinear SDE, instead of Eq. (A1) we consider

$$\frac{d\mathbf{Z}}{dt}=\mathbf{F}(\mathbf{Z},t,\mathbf{Y}) \tag{A3}$$

and suppose that F_i can be split into two parts,

$$F_i(\mathbf{Z},t,\mathbf{Y})=F_i^0(\mathbf{Z})+\alpha F_i^1(\mathbf{Z},t,\mathbf{Y}), \tag{A4}$$

where F_i^0 is stationary and not stochastic while F_i^1 is random and of zero mean. Let the function $\mathbf{y}(t)$ be a single realization of a stationary stochastic process $\mathbf{Y}(t)$. A deterministic ODE,

$$\frac{dz_i}{dt}=F_i(\mathbf{z},t,\mathbf{y}(t)), \tag{A5}$$

gives the trajectory of a particular realization $\mathbf{z}$ of the process $\mathbf{Z}$ in the phase space. Density of the flow in $\mathbf{z}$ space satisfies the Liouville equation

$$\frac{\partial\rho(\mathbf{z},t)}{\partial t}=-\left[\frac{\partial F_i^0(\mathbf{z})}{\partial z_i}+\alpha\frac{\partial F_i^1(\mathbf{z},t,y)}{\partial z_i}\right]\rho =[A_0(\mathbf{z})+\alpha A_1(\mathbf{z},t,y)]\rho, \tag{A6}$$

with the operators A_0 and A_1 introduced as

$$A_0=-\frac{\partial(F_i^0\cdot)}{\partial z_i},\quad A_1=-\frac{\partial(F_i^1\cdot)}{\partial z_i}.$$

The explicit dependence of F_i^1 and A_1 on $\mathbf{Y}(t)$ will henceforth be skipped in the notation. When $\mathbf{Y}$ is substituted for $\mathbf{y}$, Eq. (A6) becomes a linear SDE for ρ. The form of Eq. (A6) is identical to that of Eq. (A1) with $\mathbf{z}$ replaced by ρ. Thus, we can write an equivalent of Eq. (A2) as

$$\frac{\partial\langle\rho\rangle}{\partial t}=-\frac{\partial[F_i^0\langle\rho\rangle]}{\partial z_i} +\left[\alpha^2\int_0^t\left\langle\frac{\partial F_i^1(t)}{\partial z_i}e^{\tau A_0}\frac{\partial F_j^1(t-\tau)}{\partial z_j^{-\tau}}\right\rangle e^{-\tau A_0}d\tau\right]\langle\rho\rangle. \tag{A7}$$

The superscript notation represents an important point and is used as follows: let $\mathbf{z}$ denote the value (at time t) of a particular realization of the stochastic process, then $\mathbf{z}^\tau$ stands for its value at time instant $t+\tau$; in particular, $\mathbf{z}^{-t}$ is the value at the initial time. More generally, suppose that f is a function defined along the trajectory of $\mathbf{z}$. Then $f(\mathbf{z}^\tau,t+\tau)$ denotes the value of the function at time $t+\tau$ on the trajectory that passed by a particular value of $\mathbf{z}$ at t. In the LHDIA notation, it would be written as

$$f(\mathbf{z}^\tau,t+\tau)\equiv f(\mathbf{z},t|t+\tau); \tag{A8}$$

t is called the labeling time and $t+\tau$ is the measuring time. This generalized notation contains both Eulerian (for $\tau=0$) and Lagrangian descriptions.

At any value of t, the density of the flow in the phase space of $\mathbf{z}$, averaged over all possible realizations $\mathbf{y}(t)$, is equal to the probability density of $\mathbf{Z}$ (cf. [20], Lemma XVI.5.3),

$$\langle\rho(\mathbf{z},t)\rangle=P(\mathbf{z},t). \tag{A9}$$

Moreover, the flow density in $\mathbf{z}$ space verifies

$$\rho(\mathbf{z},t)=\rho(\mathbf{z}^{-t},0)\frac{D\mathbf{z}^{-t}}{D\mathbf{z}}, \tag{A10}$$

where $D\mathbf{z}^\tau/D\mathbf{z}$ stands for the Jacobian of the transformation $\mathbf{z}=\mathbf{z}(t)\to\mathbf{z}^\tau=\mathbf{z}(t+\tau)$.

Now, consider the unperturbed Liouville equation, i.e., Eq. (A6) with $\alpha=0$. It is easily verified that its solution is given by $e^{tA_0}f(\mathbf{z})$, where $f(\mathbf{z})$ is any function. Then, the following identity is obtained from Eq. (A10):

$$e^{tA_0}f(\mathbf{z})=f(\mathbf{z}^{-t})\frac{D\mathbf{z}^{-t}}{D\mathbf{z}}. \tag{A11}$$

We also note in passing that, in particular, one can make the substitution $e^{\tau A_0}=D\mathbf{z}^{-\tau}/D\mathbf{z}$ in Eq. (A7). As a consequence of Eq. (A11), we have

$$e^{-\tau A_0}\rho(\mathbf{z},t)=\rho(\mathbf{z}^\tau,t)\frac{D\mathbf{z}^\tau}{D\mathbf{z}}. \tag{A12}$$

Substituting this in Eq. (A7), accounting for Eq. (A9), and using the identity (A11) with

$$f(\mathbf{z}) = \frac{\partial F_j^1(\mathbf{z},t-\tau)}{\partial z_j} e^{-\tau A_0} \langle \rho(\mathbf{z},t) \rangle = \frac{\partial F_j^1(\mathbf{z},t-\tau)}{\partial z_j} P(\mathbf{z}^\tau,t) \frac{D\mathbf{z}^\tau}{D\mathbf{z}},$$

the final form of the transport equation for the probability density function

$$\frac{\partial P(\mathbf{z},t)}{\partial t} = -\frac{\partial [F_i^0 P(\mathbf{z},t)]}{\partial z_i} + \alpha^2 \frac{\partial}{\partial z_i} \int_0^t d\tau \left\langle F_i^1(\mathbf{z},t) \frac{D\mathbf{z}^{-\tau}}{D\mathbf{z}} \frac{\partial}{\partial z_j^{-\tau}} F_j^1(\mathbf{z}^{-\tau},t-\tau) \right\rangle \frac{D\mathbf{z}}{D\mathbf{z}^{-\tau}} P(\mathbf{z},t) \tag{A13}$$

is obtained. The first expression on the right-hand side represents the transport of the PDF by a purely deterministic operator F^0 while the second corresponds to the influence of the stochastic term which depends on correlations of the random component F^1. Moreover, it is noticed that to treat a nonlinear SDE, one has to revert to the PDF of the process, which restores linearity of the description at the expense of increasing the dimensionality of the problem, cf. Eq. (A2) versus Eq. (A13).

[1] M. R. Maxey and J. J. Riley, Phys. Fluids **26**, 883 (1983).

[2] R. Gatignol, J. Mec. Theor. Appl. **1**, 143 (1983).

[3] M. W. Reeks, Phys. Fluids A **3**, 446 (1991).

[4] M. W. Reeks, Phys. Fluids A **4**, 1290 (1992).

[5] M. W. Reeks, Phys. Fluids A **5**, 750 (1993).

[6] J. M. MacInnes and F. V. Bracco, Phys. Fluids A **4**, 2809 (1992).

[7] J.-P. Minier and J. Pozorski, EDF Internal Report No. HE-44/92.29 (1992).

[8] L. Arnold, *Stochastic Differential Equations, Theory and Applications* (Wiley, New York, 1981).

[9] R. H. Kraichnan, J. Fluid Mech. **83**, 349 (1977).

[10] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon Press, Oxford, 1990).

[11] K. E. Hyland, in *Proceedings of the 6th International Symposium on Gas-Solid Flows*, edited by D. E. Stock, Y. Tsuji, M. W. Reeks, E. E. Michaelides, and M. Gautam (American Society of Mechanical Engineers, New York, 1995), Vol. 228, p. 241.

[12] J.-P. Minier, in *Advances in Turbulence Modelling, 1997-1998 Lecture Series Programme* (Von Karman Institute, Rhode-Saint-Genese, Belgium, 1998).

[13] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1990).

[14] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1971).

[15] S. B. Pope, Annu. Rev. Fluid Mech. **26**, 23 (1994).

[16] J.-P. Minier and J. Pozorski, Phys. Fluids **9**, 1748 (1997).

[17] S. B. Pope, Phys. Fluids **6**, 973 (1994).

[18] J. Pozorski, J.-P. Minier, and O. Simonin, in *Proceedings of the 5th International Symposium on Gas-Solid Flows*, edited by D. E. Stock, Y. Tsuji, M. W. Reeks, E. E. Michaelides, and M. Gautam (American Society of Mechanical Engineers, New York, 1993), Vol. 166, p. 63.

[19] O. Simonin, E. Deutsch, and J.-P. Minier, Appl. Sci. Res. **51**, 275 (1993).

[20] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, 2nd ed. (North-Holland, Amsterdam, 1992).

[21] J. Pozorski, J. Theor. Appl. Mech. (Warsaw, Poland) **36**, 31 (1998).